Europ. J. Combinatorics (1998) **19**, 419–422
Article No. ej970197

A Representation of Even Permutations

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We give a combinatorial proof of the formula giving the number of representations of an even permutation σ in S_n as a product of an n-cycle by an $(n-2)$-cycle, such a number being $(n-\chi(\sigma))(n-3)!$, where $\chi(\sigma)$ is the number of fixed points of σ. This proof relies on the fact that any odd permutation in S_n is the product of an n-cycle by an $(n-1)$-cycle in exactly $2(n-2)!$ different ways.

Let σ be a permutation in S_n. If σ is odd, then the number of different representations of σ as a product of an n-cycle by an $(n-1)$-cycle is $2(n-2)!$. If σ is even, the number of the representations of σ as a product of an n-cycle by an $(n-2)$-cycle is $(n-\chi(\sigma))(n-3)!$, where $\chi(\sigma)$ is the number of fixed points of σ. There exist several proofs of these two formulas, relying on the theory of characters [2] or not [1]. As regards the first formula, there is also a combinatorial proof by Machì [3], which contains a recursive algorithm to construct all the above representations. In this paper, we give a combinatorial proof of the second formula, assuming the validity of the first one. Since our proof constructively reduces the problem of the representation of an even σ in S_n to the problem of the representation of an odd ρ in S_{n-1}, then if we put together our present result with the combinatorial proof of Machì, we obtain an algorithm to construct all the representations of any even σ as a product of an n-cycle by a $(n-2)$-cycle.

We adopt the following notations and definitions. We always assume $n \geq 3$. Then we let $[n] = \{1, 2, \ldots, n\}$, S_n be the the group of all the permutations on $[n]$, C_m be the conjugacy class of all the m-cycles in S_n; for any subset X of $[n]$, we let $\mathrm{Sym}\,(X)$ be the the subgroup of all the permutations in S_n which leave each element of $[n] \setminus X$ fixed, and $C_m(X)$ be the subset of all the m-cycles in C_m which leave each element of $[n] \setminus X$ fixed. We let σ be an even permutation in S_n, and we put

$$E(\sigma) = \{(\gamma, \delta) \in C_n \times C_{n-2} \mid \sigma = \gamma\delta\}.$$

For $n \geq 4$, the number $|E(\sigma)|$ is equal to the number of representations of σ as a product of an n-cycle by an $(n-2)$-cycle (for $n = 3$ see Example 1).

Note that if $\sigma = \gamma\delta$, with $(\gamma, \delta) \in C_n \times C_{n-2}$, and if y is one of the fixed points of δ, then y is necessarily not fixed by σ. This shows that if σ is the identity, then $|E(\sigma)| = 0$. Now assume that σ is not the identity. In this case, we can take some $x \in [n] \setminus \mathrm{Fix}\,(\sigma)$, we put $y = x^\sigma$ and we consider the transposition (xy). Then the permutation $\rho_x = \sigma(xy)$ is an odd permutation which leaves x fixed, hence it belongs to $\mathrm{Sym}\,([n]\setminus\{x\})$, which is isomorphic to S_{n-1}. We shall show that we can recover all the representations $\sigma = \gamma\delta$, with $(\gamma, \delta) \in C_n \times C_{n-2}$, from all the representations $\rho_x = cd$, with $(c, d) \in C_{n-1}([n] \setminus \{x\}) \times C_{n-2}([n] \setminus \{x\})$, of all the ρ_xs for $x \in [n] \setminus \mathrm{Fix}\,(\sigma)$. We put, for any $x \in [n] \setminus \mathrm{Fix}\,(\sigma)$,

$$D_x = \{(c, d) \in C_{n-1}([n] \setminus \{x\}) \times C_{n-2}([n] \setminus \{x\}) \mid \rho_x = cd\}.$$

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The number $|D_x|$ does not depends on σ and x, since the number of representations of an odd permutation of S_{n-1} as the product of an $(n-1)$-cycle by an $(n-2)$-cycle does not depend on the permutation itself, but just on n. Indeed, we have $|D_x| = 2(n-3)!$, if $n \geq 4$, and $|D_x| = 1$, if $n = 3$.

THEOREM. *Let $n \geq 4$ and σ be an even permutation in S_n. Then*

$$|E(\sigma)| = \frac{1}{2} \sum_{x \in [n] \setminus \mathrm{Fix}\,(\sigma)} |D_x|.$$

PROOF. If σ is the identity, we know that $|E(\sigma)| = 0$; on the other hand, $\mathrm{Fix}\,(\sigma) = [n]$, hence the sum is void and the formula holds true. So we assume henceforth that σ is different from the identity. Let $x \in [n] \setminus \mathrm{Fix}\,(\sigma)$ and put $y = x^\sigma$. Then $\rho_x = \sigma(xy)$ is an odd permutation in $\mathrm{Sym}\,([n] \setminus \{x\})$. Consider a representation $\rho_x = cd$, with $(c, d) \in D_x$, so that c is an $(n-1)$-cycle and d is an $(n-2)$-cycle both leaving x fixed. From this we may obtain a desired representation of σ. Indeed,

$$\sigma = \rho_x(xy) = cd(xy) = [c(xy)][d^{(xy)}].$$

Since c is an $(n-1)$-cycle fixing x, then $\gamma = c(xy)$ is an n-cycle; since d is an $(n-2)$-cycle, then $\delta = d^{(xy)}$ is an $(n-2)$-cycle too. So we have a representation $\sigma = \gamma\delta$, with $(\gamma, \delta) \in E(\sigma)$. Note that since d leaves x fixed, then y turns out to be one of the (two) fixed points of δ.

Now we show that all the representations in $E(\sigma)$ are obtained in this way. Indeed, suppose that $\sigma = \gamma\delta$, with γ an n-cycle and δ an $(n-2)$-cycle. Let y be one of the fixed points of δ. Mimicking in the reverse order the above construction, we consider the transposition (xy), where $x = y^{\sigma^{-1}}$; note that $x \neq y$, since y is not left fixed by σ. Then we have:

$$\rho_x = \sigma(xy) = \gamma\delta(xy) = [\gamma(xy)][\delta^{(xy)}].$$

It is clear that $c = \gamma(xy)$ is an $(n-1)$-cycle and that $d = \delta^{(xy)}$ is an $(n-2)$-cycle. Moreover, from $x^\gamma = x^{\sigma\delta^{-1}} = y$ it follows that c leaves x fixed, and since δ leaves y fixed then d leaves x fixed. So we have $(c, d) \in D_x$, and it is clear that (γ, δ) can be recovered from it.

Indeed, any representation in $E(\sigma)$ comes from exactly two representations (c, d). First fix some $x \in [n] \setminus \mathrm{Fix}\,(\sigma)$. It is clear that two different representations of ρ_x give different representations of σ. On the other hand, if a representation $\sigma = \gamma\delta$ derives from $(c, d) \in D_x$, then x^σ is left fixed by δ. Therefore, if $\mathrm{Fix}\,(\delta) = \{y_1, y_2\}$ and $x_1 = y_1^{\sigma^{-1}}$ and $x_2 = y_2^{\sigma^{-1}}$, we have just two possible representations, say (c_1, d_1) for ρ_{x_1} and (c_2, d_2) for ρ_{x_2}, which give rise to the same representation $\sigma = \gamma\delta$. In fact, these two representations are necessarily distinct, since the fixed points of c_1 and c_2 are distinct. □

REMARK 1. The theorem can be concisely expressed in the following way. Assume that σ is different from the identity. For each $x \in [n] \setminus \mathrm{Fix}\,(\sigma)$, define the map

$$f_x : D_x \to E(\sigma), \qquad (c, d) \mapsto (c(xx^\sigma), d^{(xx^\sigma)}).$$

Then put

$$D = \bigcup_{x \in [n] \setminus \mathrm{Fix}\,(\sigma)} D_x,$$

note that this union is disjoint, and define the map

$$f : D \to E(\sigma), \qquad (c, d) \mapsto f_x((c, d)) \quad \text{if} \quad (c, d) \in D_x.$$

Then the theorem says that for $n \geq 4$ the map f is 2–1. Moreover, it is easy to see that for $n = 3$ the map f is 3–1.

REMARK 2. The proof of the theorem contains an algorithm to find all the representations $(\gamma, \delta) \in E(\sigma)$, if we know all the representations $(c, d) \in D_x$ for all $x \in [n] \setminus \mathrm{Fix}(\sigma)$. The latter representations can be recursively constructed by the method of Machì [3].

COROLLARY. *The number of representations of an even permutation $\sigma \in S_n$ as a product of an n-cycle by an $(n-2)$-cycle is equal to $(n - \chi(\sigma))(n-3)!$.*

PROOF. For $n = 3$, the number of such representations is 0 if σ is the identity and is 3 otherwise, which agrees with $(n - \chi(\sigma))(n-3)!$ in both cases. For $n \geq 4$, the number of such representations is equal to $|E(\sigma)|$. By the theorem and by the fact that $|D_x| = 2(n-3)!$, we get

$$|E(\sigma)| = \frac{1}{2} \sum_{x \in [n] \setminus \mathrm{Fix}(\sigma)} |D_x| = \frac{1}{2}(n - \chi(\sigma))2(n-3)! = (n - \chi(\sigma))(n-3)!.$$

□

REMARK 3. An equivalent form of the corollary can be given in terms of multiplication of conjugacy classes. If we consider the following formula

$$C_n C_{n-2} = \sum_C \alpha_C C,$$

where the sum is over all the conjugacy classes of S_n, then we have

$$\alpha_C = \begin{cases} 0 & \text{if } C \text{ is odd} \\ (n - \chi(C))(n-3)! & \text{if } C \text{ is even.} \end{cases}$$

REMARK 4. We have

$$|C_n| = (n-1)! \quad \text{and} \quad |C_{n-2}| = \frac{n(n-1)}{2}(n-3)!,$$

hence

$$|C_n C_{n-2}| = \frac{n(n-1)}{2}(n-1)!(n-3)!.$$

Since $|A_n| = n!/2$, we have that the mean value of $|E(\sigma)|$ is $(n-1)(n-3)!$. Moreover, we have

$$(n-3)! \sum_{\sigma \in A_n} (n - \chi(\sigma)) = \sum_{\sigma \in A_n} |E(\sigma)| = (n-1)(n-3)! \frac{n!}{2},$$

which implies

$$\sum_{\sigma \in A_n} \chi(\sigma) = \frac{n!}{2}.$$

This formula is well known and can be easily obtained with character-theoretic methods. The point here is that now we have a pure combinatorical proof of its.

EXAMPLE 1. $n = 3$ and $\sigma = (123)$. The different representations of (123) as a product of a 3-cycle by an 1-cycle are (123)(1), (123)(2) and (123)(3). Here we consider the three 1-cycles (1), (2) and (3) as distinct, even if they all represent the identity. Moreover, note that $|E(\sigma)| = 1$, since the only element in C_1 is the identity.

EXAMPLE 2. $n = 4$ and $\sigma = (123)(4)$. Then $\chi(\sigma) = 1$ and $|E(\sigma)| = 3$. We can take $x = 1, 2$, or 3. For $x = 1$, we have $y = 2$, and $\rho_1 = (1)[(23)(4)]$. Since $(23)(4) = (234)(24) = (243)(34)$, we obtain $\sigma = (1234)(14) = (1243)(34)$. Note that in the first representation the fixed points of δ are 2 and 3, so such representations will be obtained also for $x = 2$. In the second representation the two fixed points are 1 and 2, so it will be obtained also for $x = 3$. For $x = 2$, we have $y = 3$, and $\rho_2 = (2)[(13)(4)]$. Then from $(13)(4) = (134)(14) = (143)(34)$ we, respectively, obtain $\sigma = (1234)(14) = (1423)(24)$. The first representation was already obtained in the previous step, while the second one was unknown. We can stop here, since we have already obtained all the representations of σ.

EXAMPLE 3. $n = 4$ and $\sigma = (12)(34)$. Then $\chi(\sigma) = 0$ and $|E(\sigma)| = 4$. We can take $x = 1, 2, 3$ or 4. For $x = 1$, we have $y = 2$, and $\rho_1 = (1)[(2)(34)]$. Since $(2)(34) = (234)(23) = (243)(24)$, we obtain $\sigma = (1234)(13) = (1243)(14)$. In the first representation the fixed points of δ are 2 and 4, so it will be obtained also for $x = 3$. In the second representation the two fixed points are 2 and 3, so it will be obtained also for $x = 4$. For $x = 2$, we have $y = 1$, and $\rho_2 = (2)[(1)(34)]$. Then from $(1)(34) = (134)(13) = (143)(14)$ we, respectively, obtain $\sigma = (1342)(23) = (1432)(24)$. The first representation can be obtained also for $x = 3$, while the second one also for $x = 4$. Note that we do not need to go further in our algorithm, since we have already obtained all the representations of σ.

EXAMPLE 4. $n = 7$ and $\sigma = (1234)(56)(7)$. Then $\chi(\sigma) = 1$ and $|E(\sigma)| = 6 \cdot 4! = 144$. We do not list here all the representaions of σ, but we show how the algorithm works in this quite general case. We can take $x = 1, 2, 3, 4, 5$, or 6. For $x = 1$, we have $y = 2$, and $\rho_1 = (1)[(234)(56)(7)]$. There are 48 different representations $(c, d) \in D_1$, and they give rise to 48 different representations of σ. Then we have $\rho_2 = (2)[(134)(56)(7)]$, $\rho_3 = (3)[(124)(56)(7)]$, and $\rho_4 = (4)[(123)(56)(7)]$. These three odd permutations have the same type as ρ_1, so it is simple to obtain the 48 representations of each of them by a suitable conjugation of the representations of ρ_1. Hence we obtain the corresponding representations of σ. At last, we have $\rho_5 = (5)[(1234)(6)(7)]$ and $\rho_6 = (6)[(1234)(5)(7)]$, which are of the same type, and from which we obtain all the representations $(\gamma, \delta) \in E(\sigma)$ such that 6 or 5 are left fixed by δ.

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Received 30 October 1997 and accepted 21 November 1997

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